Linear Phase FIR Filters

A system is called a linear phase system, if its group delay is a constant. In this section, four types of linear phase system will be discussed.

<table>
<thead>
<tr>
<th>Type</th>
<th>Filter Length N</th>
<th>Impulse Response Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Odd</td>
<td>Symmetric</td>
</tr>
<tr>
<td>II</td>
<td>Even</td>
<td>Symmetric</td>
</tr>
<tr>
<td>III</td>
<td>Odd</td>
<td>Anti-symmetric</td>
</tr>
<tr>
<td>IV</td>
<td>Even</td>
<td>Anti-symmetric</td>
</tr>
</tbody>
</table>

Table 1: Properties of FIR Linear Phase Filter

Note: Filter length N is equal to highest order plus 1.

In this lecture, our goal is to show that given a type I linear phase FIR filter, it is possible to synthesize type II, III and IV FIR filters by cascading zeros at $z = -1$ or $z = 1$. 
**Type I**

From the properties above, given a symmetric and odd length type I filter.

\[ h_1[n] = \delta[n] - \frac{5}{2} \delta[n - 1] + \delta[n - 2] \]

Its corresponding z transform is,

\[ H_1(z) = 1 - \frac{5}{2} z^{-1} + z^{-2} \]

\[ = (1 - \frac{1}{2} z^{-1})(1 - 2z^{-1}) \]

From the impulse response, we can observe that it is symmetric impulse response and the length of this FIR filter is 3, which is odd. Therefore, it is a Type I linear phase filter. Now, from its pole zero plot, there exists zeros located at \( z=2 \) and \( z=\frac{1}{2} \).

Figure 1: (Left) Pole Zero Plot of \( H_1(z) \), (Right) System Model of \( H_1(z) \)
Using simulink, we can simulate its frequency response.

![Simulink frequency response](image)

**Figure 2**: Impulse Response, Magnitude and Phase Response of $H_1(z)$

**Type II**

Instead of referring to properties, we add a zero $z = -1$ to $H_1(z)$. Hence,

$$H_2(z) = H_1(z)(1 + z^{-1})$$

$$= (1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 + z^{-1})$$

$$= 1 - \frac{3}{2}z^{-1} - \frac{3}{2}z^{-2} + z^{-3}$$

Mathematically, we can see that the length of $H_2(z)$ is 4, which is even. Also, its impulse response is symmetric. Therefore, we can generate a Type 2 FIR linear filter by add a zero at $z = -1$ to Type 1 FIR filter.
Figure 3: (Left) Pole Zero Plot of $H_2(z)$, (Right) System Model of $H_2(z)$

Using simulink, we can simulate its frequency response.

Figure 4: Impulse Response, Magnitude and Phase Response of $H_2(z)$
Type III

Now, we can also add a zero, \( z = 1 \), to \( H_2(z) \) and formulate a type III linear phase filter.

\[
H_3(z) = H_2(z)(1 - z^{-1})
\]

\[
= (1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 + z^{-1})(1 - z^{-1})
\]

\[
= 1 - \frac{5}{2}z^{-1} + \frac{5}{2}z^{-3} - z^{-4}
\]

From, Eq.(8), the impulse response if anti-symmetric and the length of \( H_3(z) \) is 5, which is odd. Therefore, it is a type III linear phase FIR filter.

Figure 5: (Left) Pole Zero Plot of \( H_3(z) \), (Right) System Model of \( H_3(z) \)

Using simulink, we can simulate its frequency response.
Similar to previous method, we can add a zero, \( z = 1 \), to \( H_1(z) \) and formulate a type IV linear phase filter.

\[
H_3(z) = H_1(z)(1 - z^{-1})
\]

\[
= (1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})(1 - z^{-1})
\]

\[
= 1 - \frac{7}{2}z^{-1} + \frac{7}{2}z^{-2} - z^{-3}
\]

From, Eq.(11), the impulse response is anti-symmetric and the length of \( H_3(z) \) is 4, which is even. Therefore, it is a type IV linear phase FIR filter.

Using simulink, we can simulate its frequency response.

Figure 6: Impulse Response, Magnitude and Phase Response of \( H_3(z) \)
In this section, we will explore the relation between maximum phase system and minimum phase system. In addition, we will have a short discussion on the impulse response of minimum phase and maximum phase systems.
Given a minimum phase IIR system,

\[ H_{\text{min}}(z) = \frac{1 - \frac{5}{6} z^{-1}}{1 - \frac{1}{2} z^{-1}} \]

From its pole zero plot, we can see that poles and zeros are all inside the unit circle and it is a minimum phase system. Using simulink, we can simulate this system and its impulse response, magnitude and phase response are shown as followed.

Figure 9: (Left) Pole zero plot of \( H_{\text{min}}(z) \), (Right) System Model of \( H_{\text{min}}(z) \)

Figure 10: (Left) Impulse response of \( H_{\text{min}}(z) \), (Right) Frequency Response of \( H_{\text{min}}(z) \)
As we learned before, we can cascade this minimum phase system with an allpass filter and we formulate an maximum phase system. The all pass filter is,

\[ A(z) = \frac{-\frac{5}{6} + z^{-1}}{1 - \frac{5}{6} z^{-1}} \]

Hence, the maximum phase system is,

\[ H_{\text{max}}(z) = H_{\text{min}}(z) A(z) \]

\[ = \frac{(1 - \frac{5}{6} z^{-1}) (-\frac{5}{6} + z^{-1})}{(1 - \frac{1}{2} z^{-1}) (1 - \frac{5}{6} z^{-1})} \]

\[ = \frac{-\frac{5}{6} + z^{-1}}{1 - \frac{1}{2} z^{-1}} \]

\[ = -\frac{5}{6} \frac{(1 - \frac{6}{5} z^{-1})}{(1 - \frac{1}{2} z^{-1})} \]

It can be seen that there exists a From the equations above, we can simulate the maximum phase system. Its impulse response, magnitude and phase response are shown as followed.

Figure 11: (Left)Pole zero plot of \( H_{\text{max}}(z) \), (Right)System Model of \( H_{\text{max}}(z) \)
From our simulation, we explored the relation between maximum phase and minimum phase systems. Furthermore, by observing the impulse response of these two systems, we can see that the minimum phase system has smaller response time but the maximum phase system has larger response time.

Figure 12: (Left) Impulse response of $H_{\text{max}}(z)$, (Right) Frequency Response of $H_{\text{max}}(z)$