THEORY AND DESIGN OF SHIFT-INVARIANT FILTER BANKS AND WAVELETS

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ABSTRACT

A drawback of critical-sampling multirate system is its shift-variant property at the subband output. This prevents wavelets from many applications where shift-invariance is required. For a given set of filter coefficients and cost function, all of the existing methods solve the problem by finding the path in the decomposition tree that minimizes shift-variance with respect to a given cost function. This procedure is signal dependent and is inefficient, especially for long data sets and images, since the subband decomposition has to be performed for all shifts of input signal during processing time. In this paper, we establish a framework for shift-invariant filter bank by connecting the relation between the polyphase representation and shift-invariant property of filter banks. Theory, analysis, and design will be presented, and comparison to the existing systems will be discussed.

1. INTRODUCTION

In a multirate system, input signal is decomposed into subbands by a set of bandpass analysis filters $H_k(z)$, which decorrelates the input signal to locate signal features in subbands. The advantages of signal decorrelation and multiresolution representation are often offset by the time-varying nature of downsampling. A shifted input $x(n-\ell_s)$ does not produce shifted subband coefficients unless ℓ_s is a multiple of M. This is called shift-variance.

Shift-invariant property is required in many signal processing applications, such as detection, image coding, and denoising. The importance of shift-invariant multirate system has been noticed by a number of researchers and several approaches have been proposed. In [1, 2], the critical sampling condition in subband system is relaxed in order to obtain shift-invariant system. Besides the increased implementation cost, the number of local extrema is unknown before detection and it is possible to locate more local extrema than subband samples obtained from the traditional filter banks with downsampling. As a result, this method is inapplicable in detection.

An alternative approach is based on the best-basis-selection method [3, 4, 5]. A cost function, Shannon's entropy, is used in those methods as a measure of shift-invariant property. The input signal is decomposed using the traditional filter bank and the decomposition is repeated for the input signal with all possible shifts. The set of subband coefficients that minimizes the given cost function is then chosen for reconstruction. It should be noticed that the system itself is not shift-invariant in this manner, and thus it is an *ad hoc* technique and the performance and resulting structure depends on input signal. Better shift-invariant property is achieved by sacrificing the computational efficiency because of the additional decomposition for each level. For example, computational complexity is $O(N^2 logN)$ for an $N \times N$ image. As a result, this approach is unsuitable for real-time processing.

The above methods try to obtain shift-invariant system by minimizing the shift-varying effect of the existing multirate system with respect to a given cost function. In fact, the problem always exists unless the filter bank itself is shift-invariance. It is our objective in this paper to provide a framework for the theory and design of shift-invariant filter banks. We propose the definition of shift-invariance in terms of the polyphase representation, derive the relation between the polyphase representation and shift-invariant property of filter banks, and propose design methods for different classes of near shift-invariant (NSI) perfect reconstruction (PR) filter banks (FB) and perfect SI (PSI) near PR (NPR) filter banks. Design examples and simulations on image coding are presented.

2. THEORY OF SI FILTER BANKS

2.1. Polyphase Components and Shift-Invariance

Let $H_k(z), k = 0, 1, ..., M-1$ be a set of FIR analysis filters in an M-channel filter banks. Let $E_{k\ell}$ be the ℓ-*th* polyphase component of $H_k(z)$, $H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_{k\ell}(z^M)$, $k = 0, 1, ..., M-1$. $\mathrm{E}(z) = [E_{k\ell}(z)]$ is the corresponding polyphase matrix [6]. After downsampling by M, only one out of M polyphase components will be retained. When the input signal is an impulse with a shift s, the ℓ_s-*th* polyphase components $E_{k,\ell_s}(z^M)$ is retained with $l_s = s \ mod \ (M)$. With this observation and the fact that multirate system is linear, we can define shift-invariance (SI) in terms of polyphase components as follows,

Definition 1 *An M-channel filter bank is k-th bank shift-invariant if the polyphase components of $H_k(z)$, $0 \le k \le M-1$, satisfy $E_{kn}(z^M) = E_{km}(z^M)$ $\quad 0 \le n, m \le M-1$.*

It should be noted that the shift-invariant property defined here is different from that in the linear system theory where SI means that when the input is shifted, the output will be shifted by the same amount. In this paper, however, shift-invariance implies that the subband signal do not change when the input signal is shifted. It can also be

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called translation-invariance as in [3, 4, 5]. The above definition is a strong SI condition. A weak form of an SI system can be defined using an additive cost function as follows:

Definition 2 *A map μ from a discrete sequences x_i to $\Re$ is called an additive cost function if $\mu(0) = 0$ and $\mu(x_i) = \sum(x_i)$.*

Definition 3 *With respect to an additive cost function μ, an M-channel filter bank is k-th bank near (or optimal) shift-invariant if $\mu(E_{kn}(z^M), E_{km}(z^M)) = \mu_{min}$ $\ 0 \leq n, m \leq M-1,\ \ 0 \leq k \leq M-1$.*

Since many applications also require PR property, the following theorem explores the possibility to obtain a system with both PR and SI properties.

Theorem 1 *In an M-channel maximally decimated filter bank, PSI and PR properties cannot be achieved simultaneously in $H_k(z)$ except for*

$$H_k(z) = 1 + z^{-1} + ... + z^{-M+1}\ . \tag{1}$$

From the above theorem, the only PSI PR filter bank is the Haar wavelet and (2,N) wavelets with $2 + N$ being the multiple of 4 for two-channel systems and (1) for M-channel systems. Since both the Haar filter bank and the filter in (1) do not have good frequency response, one needs to sacrifice either property to obtain a good filter bank. We focus our discussion on the SI property of lowpass filter $H_0(z)$. The theory and design can be easily extended to the other channel or channels.

To quantify the SI property of a filter bank, the group delay is used in all SI filter design in this paper. The reason for this choice is that the polyphase components are the delayed and decimated versions of the input impulse response. The smaller the difference in group delays, the better the phases alignment, which implies small shift-variance. The above observation can be concluded by the following theorem

Theorem 2 *The shift-variance of a filter bank is proportional to the difference of the group delay between the polyphase components of an analysis filter. When all the polyphase components of a given analysis filter have the same group delay, the filter is perfect shift-invariant.*

2.2. Analysis of Linear-Phase and/or Paraunitary Filter Banks

Lemma 1 *Consider an analysis filter $H_k(z)$ in an M-channel LP filter bank with even M,*

- *Odd length N: There are two linear phase polyphase components. The other $M-2$ polyphase components are pairwise time-reversal.*
- *Even length N: M polyphase components are either all pairwise time-reversal or all linear phase with the same length.*

Lemma 2 *For M-channel LP filter banks with even M and even length, perfect SI can be achieved if all polyphase components are LP.*

Lemma 3 *Consider a given analysis filter $H_k(z)$ in an M-channel LP filter bank with odd M,*

- *Even length N: There exists one LP polyphase component and the other $M-1$ polyphase components are pairwise time-reversal.*
- *Odd length N: The polyphase components are either the same as in the case of even length or all linear phase with the same length.*

Lemma 4 *For M-channel LP filter banks with odd M and odd length, perfect SI can be achieved if all polyphase components are LP.*

All polyphase components of a given analysis filter $H_k(z)$ in an M-channel paraunitary filter bank are not linear phase. Therefore, to obtain an NSI M-channel paraunitary filter bank, one has to minimize the differences of group delay between all polyphase components $E_{k,l}(z)$.

Lemma 5 *In a two-channel odd length LP filter bank (type B), the difference of group delay between the two polyphase components of $H_k(z)$ is a constant. In other words, the SI property cannot be improved for a single level.*

According to the above theorem, minimizing the group delay between the polyphase components of a multi-level analysis filter instead of a single level has to be considered for the design of odd length LP NSI filter bank.

Lemma 6 *SI property of even-length LP two-channel filter banks can be achieved or improved in a single level.*

For a two-channel paraunitary filter bank, it is always possible to obtain better SI property by minimizing the differences of group delay between the polyphase components in a single level.

Property 1: If the lowpass filter in a two-channel filter bank possesses SI property, the analysis scaling function of dyadic decomposition using this filter bank also have SI property.

Property 2: If an M-channel filter bank possesses SI property, the analysis scaling function of the resulting M-band wavelets is also SI.

3. SHIFT-INVARIANT FILTER BANK DESIGN

In [3, 4], the cost function is calculated for all possible shifts of the input signal each time the algorithm is applied. The cost function in our method is a function of the filter coefficients and used in filter design. Once the coefficients are found, the resulting system is SI for all shift of the input signal, that is input-independent, and no cost function is needed to calculate during processing time. The difference of group delay is chosen as the cost function for SI optimization. When the difference in group delay is minimized and the unified coding gain [6] is maximized, the design can then be formulated as

$$\min \quad \left\{ p \sum_i \int ||D^i_{gpdelay}(\omega)|| d\omega \right\} + \max\{p_1 G_{SBC}\}$$
$$\text{subject to} \begin{cases} \frac{d^n}{dz^n} H_0(z)|_{z=-1} = 0 \\ \text{PR constraints} \\ \text{other constraints} \end{cases} \tag{2}$$

where $||\cdot||$ represents norm, which can be l^1, l^2, or l^∞, $D_{gddelay}$ is the difference between group delays, G_{SBC} is the unified coding gain, and $\frac{d^n}{dz^n}H_0(z)|_{z=-1} = 0$ is used to obtain high regularity.

NSI PR Linear Phase Filter Banks: Two types of non-trivial LP two-channel filter banks can be parameterized by lattice structures [7, 8]. The advantage of lattice structure is that both the LP and PR properties are structurally imposed, independent of the choice of lattice coefficients. NSI PR filter bank design using lattice structure can be summarized as follows [9]:

- For a given set of lattice coefficients, calculate $H_0(z)$ using corresponding lattice structure.
- Calculate the difference of group delay between the polyphase components. This yields $\int ||D_{gpdelay}(\omega)||d\omega$.
- Compute the unified coding gain, which gives G_{SBC} in (2), or other objective functions.
- Perform the above steps iterately to find the lattice coefficients that minimize (2).

The frequency responses and scaling functions of a design example with $N_{H_0} = 9, N_{H_1} = 7$ are shown in Fig. 1. Similar plots for $N_{H_0} = 10, N_{H_1} = 10$ are shown in Fig. 2.

As a design example for M-channel NSI PR LP pairwise mirror-image (PMI) FB, a 3-channel example with $N_{H_0} = N_{H_2} = 56$ and $N_{H_1} = 53$ using lattice structure proposed in [10] is shown in Fig. 3. Coding gain and the stopband attenuation are used as additional cost functions.

Paraunitary NSI PR Filter Banks: Lattice structure in [11] is used for 2-channel paraunitary NSI PR filter bank design. A design example for a 3-channel paraunitary NSI PR PMI filter banks with $N_{H_0} = N_{H_1} = N_{H_2} = 20$ is shown in Fig. 5.

PSI NPR Filter Banks: PR and SI cannot be achieved at the same time except for the special cases in Theorem 1. By allowing some reconstruction error, one can design PSI system. Lattice structure cannot be used here, therefore we adopt the time domain design method in [12] for the design of PSI NPR filter banks.

To achieve PSI property, $H_0(z)$ in a two-channel system must have the following impulse response

$$\mathbf{h}_0 = [a\ a\ b\ b\ c\ c\ \cdots\ e\ e\ \cdots\ c\ c\ b\ b\ a\ a]. \tag{3}$$

$H_0(z)$ and $F_1(z)$ are used to form a modified polyphase matrix

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_0(0) & \mathbf{E}_1(0) \\ \mathbf{E}_0(1) & \mathbf{E}_1(1) \\ \vdots & \vdots \\ \mathbf{E}_0(N/2) & \mathbf{E}_1(N/2) \end{bmatrix}$$

where $\begin{aligned} \mathbf{E}_0(i) &= [h_0(i)\ \ h_0(i+2)\ \ \cdots\ \ h_0(N/2+i-1)] \\ \mathbf{E}_1(i) &= [f_1(i)\ \ f_1(i+2)\ \ \cdots\ \ f_1(N/2+i-1)] \end{aligned}$,

and N is the filter length. $H_1(z)$ and $F_0(z)$ can be obtained by inverting the modified polyphase matrix $\mathbf{R}^T = \mathbf{E}^{-1}$. The resulting filter banks will be NPR and PSI. A design example with $N_{h0} = N_{h1} = 14$ is shown in Fig. 6.

4. SIMULATION RESULTS

In our simulation, DWT and an uniform scalar quantizer are used in the first two stages of a wavelet-based image coder. A sequential baseline coding method similar to the JPEG standard is applied to the scanned coefficients [13]. The image compression results using Daubechies (9,7) tap FB and SI (9,7) tap FB on "boats" image are shown in Fig. 7. The filter banks have compatible objective performance, however, the perceptual quality of the SI system is better than that of the D(9,7) system.

5. CONCLUSION

We established a framework for shift-invariant filter bank design for the first time. Relating the polyphase representation and shift-invariant property of filter banks is one of the contributions in this paper. SI filter bank design methods by minimizing the difference in group delay was proposed. By this method, SI system can be obtained without searching decomposition path during processing time and changing the structure of traditional multirate system. Design examples and simulation results were presented. Our recent works have been focused on the framework of two-dimensional SI FB design [14] and continuous time SI multiresolution analysis [15].

6. REFERENCES

[1] S. Mallat and W. Hwang, "Singularity detection and processing with wavelets", *IEEE Trans. on IT*, vol. 38, pp. 617–643, 1992.

[2] S. Mallat and S Zhong, "Characterization of signals from multiscale edges", *IEEE Trans. on PAMI*, vol. 14, pp. 710–732, July 1992.

[3] J. Liang and T. W. Parks, "A two-dimensional translation invariant wavelet representation and its applications", in *Proc. of IEEE ICIP*, 1994, vol. 1, pp. 66–69.

[4] S. D. Marco, P. N. Heller, and J. Weiss, "An M-band, 2-dimensional translation-invariant wavelet transform and applications", in *Proc. of IEEE ICASSP*, 1995, pp. 1077–1080.

[5] J. Weiss, "Translation invariance and the wavelet transform", Tech. Rep., Aware, Inc., 1994.

[6] P. P. Vaidyannathan, *Multirate systems and filter banks*, Prentice Hall, 1993.

[7] T. Q. Nguyen and P. P. Vaidyannathan, "Two-channel perfect reconstruction FIR QMF structures which yield linear-phase analysis and synthesis filters", *IEEE Trans. on ASSP*, vol. 37, pp. 676–690, 1989.

[8] G. Strang and T. Q. Nguyen, *Wavelets and filter banks*, Wellesley-Cambridge, 1996.

[9] Y. Hui, C. W. Kok, and T. Q. Nguyen, "Shift-invariant filter banks: Theory, design, and applications", *submitted to IEEE Trans. on SP*, Feb. 1996.

[10] T. Q. Nguyen and P. P. Vaidyannathan, "Maximally decimated perfect-reconstruction FIR filter banks with pairwise mirror-image analysis (and synthesis) frequency responses", *IEEE Trans. on ASSP*, vol. 36, pp. 693–705, May 1988.

[11] P. P. Vaidyannathan and P.-Q Hoang, "Lattice structure for optimal design and robust implementation of two-channel perfect reconstruction QMF banks", *IEEE Trans. on Acoust., Speech, Signal Processing*, vol. 36, pp. 81–94, Jan. 1988.

[12] C. W. Kok and T. Q. Nguyen, "Discrete coefficient filter banks and applications in image coding", in *Proc. of IEEE ICASSP*, 1996.

[13] S. Trautmann and T. Q. Nguyen, "Comparison of linear-phase perfect-reconstruction filter banks in wavelet-transform-based image compression", in *Proc. of Conf. on Information Sci. and Sys.*, 1995.

[14] Y. Hui, C. W. Kok, and T. Q. Nguyen, "Two-dimensional shift-invariant filter banks: Theory and design", in *to be presented on SPIE'96*, Aug. 1996.

[15] C. W. Kok, Y. Hui, and T. Q. Nguyen, "Translation invariant multiresolution analysis: Theory and applications", *to be submitted to IEEE Trans. on SP*, 1996.

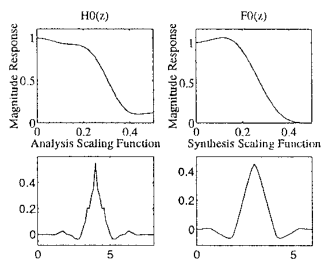

Figure 1: A 2-channel NSI PR LP FB with (9,7) tap using 3-level optimization.

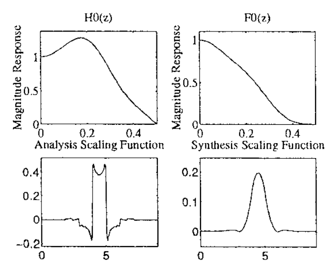

Figure 2: A 2-channel NSI PR LP FB with (10,10) tap.

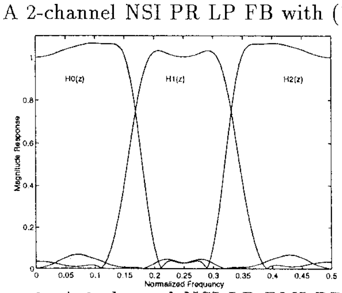

Figure 3: A 3-channel NSI LP PMI PR FB.

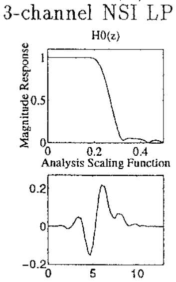

Figure 4: A 2-channel near SI paraunitary PR FB with (14,14) tap.

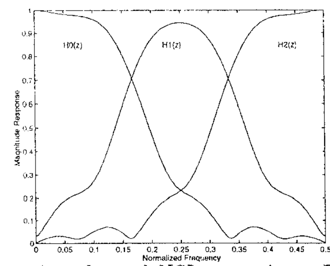

Figure 5: A 3-channel NSI paraunitary PMI FB.

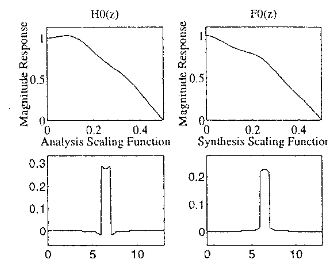

Figure 6: A 2-channel PSI NPR FB with (14,14) tap.

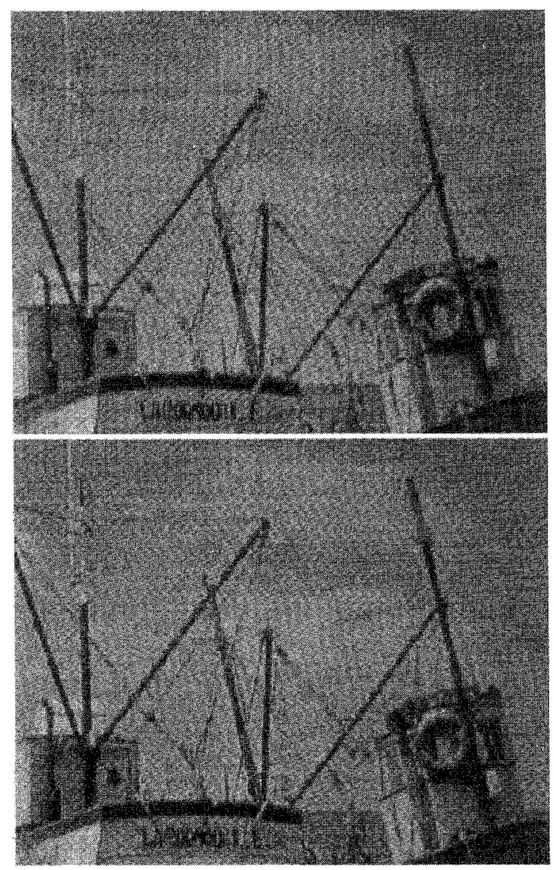

Figure 7: Reconstructed "boats" image using D(9,7) and SI(9,7) at 0.4 bpp.