A CLASS OF GENERALIZED COSINE-MODULATED FILTER BANK [1]

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Abstract - It is well known that FIR filter banks which satisfy the perfect reconstruction (PR) property, can be obtained by cosine modulation of a linear-phase prototype filter of length $N = 2mM$, where M is the number of channels. Moreover, the overall delay of this system is $(2mM - 1)$ samples. In this paper, we present a PR cosine-modulated filter bank in which its overall delay is $(2(\alpha + 1)M - 1)$ samples, where α is a positive integer in the range $0 \leq \alpha \leq 2m - 2$. Consequently, as will be shown, the prototype filter of the proposed filter bank no longer has linear phase.

I. Introduction

Fig. 1 shows a typical M-channel maximally-decimated parallel filter bank where $H_k(z)$ and $F_k(z)$, $0 \leq k \leq M - 1$, are the analysis and synthesis filters, respectively (we consider only finite impulse response (FIR) filters in this paper). The theory for perfect reconstruction has been established [1-4]. Recently, the perfect-reconstruction (PR) cosine-modulate filter bank has emerged as an optimum filter bank with respect to implementation cost and design ease [4]. The impulse responses of the analysis and synthesis filters, $h_k(n)$ and $f_k(n)$, are cosine-modulated versions of the prototype filters $h(n)$ and $f(n)$, respectively [4]. In other words,

$$\begin{cases} h_k(n) = 2h(n)\cos\Big((2k+1)\frac{\pi}{2M}(n - \frac{N-1}{2}) + \theta_k\Big), \\ f_k(n) = 2f(n)\cos\Big((2k+1)\frac{\pi}{2M}(n - \frac{N-1}{2}) + \gamma_k\Big), \end{cases} \quad (1)$$

where $0 \leq n \leq N - 1$, $0 \leq k \leq M - 1$ and

$$\theta_k = (-1)^k \frac{\pi}{4}, \quad \gamma_k = -(-1)^k \frac{\pi}{4}. \quad (2)$$

Here, the lengths of $H_k(z)$ and $F_k(z)$ are assumed to be multiples of $2M$, i.e., $N = 2mM$. Clearly from the above equation, the only parameters to be found are $h(n)$ and $f(n)$, $0 \leq n \leq N - 1$. Moreover, if the cosine-modulated filter bank is lossless and if $h(n)$ and $f(n)$ are linear-phase, then the impulse responses are related as follows [4]:

$$f(n) = h(n), \quad f_k(n) = h_k(N - 1 - n). \quad (3)$$

Thus, if the cosine-modulated filter bank is lossless, and if the prototype filters are linear-phase, then the only unknown parameters are $h(n)$ [4]. Furthermore, the overall delay of the system is $(N - 1)$ samples, i.e., $\hat{x}(n) = x(n - (N - 1))$. In other words, the output signal is valid at and after the time

$$T_v = (N - 1)T_s = (2mM - 1)T_s, \quad (4)$$

where T_s is the sampling time of the input signal. In applications where the overall delay T_v is desired to be small, the above cosine-modulated filter bank (lossless and linear-phase) might not be appropriate. It is judicious to ask the following question: Does a PR cosine-modulated filter bank in which its overall delay is smaller than the above T_v in (4), exist? The paper shows that the answer is affirmative.

In this paper, we consider the design of the PR cosine-modulated filter bank where the overall delay time T_v is a variable. Specifically, T_v can take any of the following values:

$$T_v = (2(\alpha + 1)M - 1)T_s, \quad 0 \leq \alpha \leq 2(m - 1). \quad (5)$$

Note that the value of T_v in (4) is one of the values in (5), i.e., $\alpha = m - 1$. As derived in the paper, the prototype filters, $H(z)$ and $F(z)$, of the variable-overall-delay cosine-modulated filter bank are equal, i.e., $H(z) = F(z)$. However, they are no longer linear phase. Section II describes the new cosine-modulated filter bank and presents an example.

Notations Bold faced letters indicate vectors and matrices. Superscript T denotes transposition and the tilde accent on a function $\mathbf{F}(z)$ is defined such that $\tilde{\mathbf{F}}(z) = \mathbf{F}_*^T(z^{-1})$, $\forall z$, where the asterisk $(*)$ subscript denotes the conjugation of coefficients. Moreover, $\mathbf{I}_M$ stands for the $M \times M$ identity matrix and $\mathbf{J}_M = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}_{M \times M}$. $[\mathbf{C}]_{k,\ell}$ denotes the (k, ℓ) element of $\mathbf{C}$.

II. The variable-overall-delay cosine-modulated filter bank

Let $\mathbf{E}(z)$ and $\mathbf{R}(z)$ be the polyphase transfer matrices of the analysis and synthesis filter banks in Fig. 1. Furthermore, let $H(z) = \sum_{\ell=0}^{2M-1} z^{-\ell} G_\ell(z^{2M})$, and $F(z) = \sum_{\ell=0}^{2M-1} z^{-\ell} K_\ell(z^{2M})$ be the prototype filters of length $N = 2mM$ and $G_\ell(z)$ and $K_\ell(z)$ are the corresponding polyphase filters of length m, respectively. It is shown in [4] that $\mathbf{E}(z)$ and $\mathbf{R}(z)$ can be expressed in terms of $G_\ell(z)$, $K_\ell(z)$ and $\hat{\mathbf{C}}$ as follows

$$\begin{cases} \mathbf{E}(z) = \hat{\mathbf{C}} \begin{pmatrix} \mathbf{g}_0(-z^2) \\ z^{-1}\mathbf{g}_1(-z^2) \end{pmatrix}, \\ \mathbf{R}(z) = \begin{pmatrix} z^{-1}\mathbf{J}_M\mathbf{k}_1(-z^2)\mathbf{J}_M & \mathbf{J}_M\mathbf{k}_0(-z^2)\mathbf{J}_M \end{pmatrix} \hat{\mathbf{C}}^T, \end{cases} \quad (6)$$

where

[1] This work was sponsored by the Department of Defense and the Department of the Air Force under contract number F19628-90-C-0002

0-7803-0593-0/92 $3.00 1992 IEEE

$$\begin{cases} \mathbf{g}_0(z) = \text{diag } (G_0(z) \quad G_1(z) \quad \dots \quad G_{M-1}(z)), \\ \mathbf{g}_1(z) = \text{diag } (G_M(z) \quad G_{M+1}(z) \quad \dots \quad G_{2M-1}(z)), \\ \mathbf{k}_0(z) = \text{diag } (K_0(z) \quad K_1(z) \quad \dots \quad K_{M-1}(z)), \\ \mathbf{k}_1(z) = \text{diag } (K_M(z) \quad K_{M+1}(z) \quad \dots \quad K_{2M-1}(z)), \end{cases} \tag{7}$$

$$[\hat{\mathbf{C}}]_{k,\ell} = 2\Big((2k+1)\frac{\pi}{2M}(\ell - \frac{N-1}{2}) + (-1)^k\frac{\pi}{4}\Big).$$

Using the above $\mathbf{E}(z)$ and $\mathbf{R}(z)$, one obtains the expression for $\mathbf{P}(z) \triangleq \mathbf{R}(z)\mathbf{E}(z)$ as follows

$$\mathbf{P}(z) = (z^{-1}\mathbf{J}_M\mathbf{k}_1(-z^2)\mathbf{J}_M \quad \mathbf{J}_M\mathbf{k}_0(-z^2)\mathbf{J}_M)\hat{\mathbf{C}}^T\hat{\mathbf{C}}\begin{pmatrix} \mathbf{g}_0(-z^2) \\ z^{-1}\mathbf{g}_1(-z^2) \end{pmatrix}. \tag{8}$$

For a PR filter bank, $\mathbf{P}(z)$ must have the form [2]

$$\mathbf{P}(z) = \begin{pmatrix} \mathbf{0} & z^{-\nu}\mathbf{I}_1 \\ z^{-(\nu+1)}\mathbf{I}_2 & \mathbf{0} \end{pmatrix} \tag{9}$$

where ν is a positive integer and the dimensions of $\mathbf{I}_1$ and $\mathbf{I}_2$ are added up to M. It is our objective to simplify (8) in order to obtain the necessary and sufficient conditions on $\mathbf{k}_\ell(z)$ and $\mathbf{g}_\ell(z), \ell = 0, 1$ for perfect reconstruction. Using the following identity from [4],

$$\hat{\mathbf{C}}^T\hat{\mathbf{C}} = \begin{pmatrix} \mathbf{I}_M - (-1)^m\mathbf{J}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M + (-1)^m\mathbf{J}_M \end{pmatrix}. \tag{10}$$

(8) becomes

$$\mathbf{P}(z) = z^{-1}\Big\{\Big[\mathbf{J}_M\mathbf{k}_1(-z^2)\mathbf{J}_M\mathbf{g}_0(-z^2) + \mathbf{J}_M\mathbf{k}_0(-z^2)\mathbf{J}_M\mathbf{g}_1(-z^2)\Big]$$
$$+(-1)^m\mathbf{J}_M\Big[-\mathbf{k}_1(-z^2)\mathbf{g}_0(-z^2) + \mathbf{k}_0(-z^2)\mathbf{g}_1(-z^2)\Big]\Big\}. \tag{11}$$

Using the forms of $\mathbf{J}_M$ and $\mathbf{I}_M$, the above equation has the following form:

$$\mathbf{P}(z) = z^{-1}\left\{\begin{pmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{pmatrix} + (-1)^m\begin{pmatrix} 0 & 0 & \dots & x \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & x & \dots & 0 \\ x & 0 & \dots & 0 \end{pmatrix}\right\}$$
$$= z^{-1}\mathbf{L}(-z^2) \tag{12}$$

where x is an arbitrary polynomial. Since we are only interested in a PR cosine-modulated filter bank, $\mathbf{L}(z)$ must be $z^{-s}\mathbf{I}_M$ where s is an arbitrary positive integer [2]. Using the form in (12) and the requirement that $\mathbf{L}(z)$ is an identity matrix, one concludes that

$$\begin{cases} \mathbf{J}_M\mathbf{k}_1(z)\mathbf{J}_M\mathbf{g}_0(z) + \mathbf{J}_M\mathbf{k}_0(z)\mathbf{J}_M\mathbf{g}_1(z) = (-z)^{-s}\mathbf{I}_M, \\ \mathbf{k}_1(z)\mathbf{g}_0(z) = \mathbf{k}_0(z)\mathbf{g}_1(z). \end{cases} \tag{13}$$

The above equations are the necessary and sufficient conditions on the polyphase filters $G_\ell(z)$ and $K_\ell(z)$, respectively for perfect reconstruction. These conditions are general versions of those reported in [4]. Specifically, suppose that $H(z)$ is linear-phase and $F(z) = z^{-(N-1)}H(z^{-1})$ as in [4], then

$$\mathbf{k}_\ell(z) = \mathbf{g}_\ell(z), \quad \mathbf{J}_M\mathbf{g}_1(z)\mathbf{J}_M = \widetilde{\mathbf{g}_0}(z), \quad \ell = 0, 1. \tag{14}$$

Substituting (14) into (13) and simplifying, one obtains

$$\begin{cases} \widetilde{\mathbf{g}_0}(z)\mathbf{g}_0 + \widetilde{\mathbf{g}_1}(z)\mathbf{g}_1 = (-z)^{-s}\mathbf{I}_M, \\ \mathbf{g}_0\mathbf{g}_1 = \mathbf{g}_1\mathbf{g}_0, \end{cases} \tag{15}$$

which is the same as in [4]. In summary, the PR cosine-modulated filter bank in [4] is a special case of the variable-overall-delay PR cosine-modulated filter bank discussed here. To be more precise, if the overall delay of the filter bank is $(N-1) = (2mM-1)$, then both filter banks are the same.

In terms of the polyphase filters $G_\ell(z)$ and $K_\ell(z)$, the conditions expressed by (13) become

$$\begin{cases} G_\ell(z)K_{2M-1-\ell}(z) + G_{M+\ell}(z)K_{M-1-\ell}(z) = (-z)^{-s}, \\ G_\ell(z)K_{M+\ell}(z) = G_{M+\ell}(z)K_\ell(z), \end{cases} \tag{16}$$

for ℓ in the range $0 \le \ell \le M-1$. The above equations imply that $G_\ell(z)$ and $K_\ell(z)$ are related, i.e., given $G_\ell(z)$, it is possible to solve for $K_\ell(z)$. Writing (16) in the matrix form, one obtains

$$\begin{pmatrix} \mathbf{T}_0(z) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{T}_{M-1}(z) \end{pmatrix}\begin{pmatrix} \mathbf{S}_0(z) \\ \vdots \\ \mathbf{S}_{M-1}(z) \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \vdots \\ \mathbf{a} \end{pmatrix} \tag{17}$$

where

$$\mathbf{T}_\ell(z) = \begin{pmatrix} G_{2M-1-\ell}(z) & G_{M-1-\ell}(z) \\ -G_{M+\ell}(z) & G_\ell(z) \end{pmatrix}, \quad \mathbf{S}_\ell(z) = \begin{pmatrix} K_\ell(z) \\ K_{M+\ell}(z) \end{pmatrix}, \tag{18}$$

and $\mathbf{a} = (1 \quad 0)^T$. Solving for $K_\ell(z)$ from (17) yields

$$\begin{pmatrix} K_\ell(z) \\ K_{M+\ell}(z) \end{pmatrix} = \frac{1}{D_\ell(z)}\begin{pmatrix} G_\ell(z) \\ G_{M+\ell}(z) \end{pmatrix}, \quad 0 \le \ell \le M-1 \tag{19}$$

where

$$D_\ell(z) = (\text{Determinant of } \mathbf{T}_\ell(z))(-z)^{-s}$$
$$= [G_\ell(z)G_{2M-1-\ell}(z) + G_{M-1-\ell}(z)G_{M+\ell}(z)](-z)^{-s}. \tag{20}$$

In summary, the polyphase filters of the new filter bank are the same except for a scale factor $D_\ell(z)$. Since we are only interested in FIR filter banks, $D_\ell(z)$ must be a delay, i.e., $D_\ell(z) = \beta_\ell z^{-\alpha}$ where β_ℓ and α_ℓ are a nonzero constant and a positive integer, respectively. Even though the scale factor β_ℓ and the delay α_ℓ can be different for different values of ℓ, we only consider the case where $\beta_\ell = 1$ and $\alpha_\ell = \alpha$ for all ℓ in this paper. Consequently, with the above choice of β_ℓ and α_ℓ, $F(z) = H(z)$ as in [4]. Moreover, the phase of $H(z)$ will be determined by the choice of α (as will be elaborated later). $D_\ell(z)$ must satisfy:

$$[G_\ell(z)G_{2M-1-\ell}(z) + G_{M-1-\ell}(z)G_{M+\ell}(z)](-z)^{-s} = z^{-\alpha}, \tag{21}$$

for ℓ in the range $0 \le \ell \le \lfloor\frac{M+1}{2}\rfloor$. Since each polynomial $G_\ell(z)$ in (21) has length m, (21) yields $(2m-1)$ conditions. Thus, if one fixes any two polynomials, for instance, $G_\ell(z)$ and $G_{M-1-\ell}(z)$, then the other two can be found ($2m$ unknowns) if s and α are known. We will elaborate on the solution of (21) in the subsection below.

The conventional cosine-modulated filter bank [4] where the prototype filter $H(z)$ and $F(z)$ are linear-phase functions can be derived from (21) for $\alpha = s = (m-1)$, i.e.,

$$G_\ell(z)G_{2M-1-\ell}(z) + G_{M-1-\ell}(z)G_{M+\ell}(z) = (-1)^{(m-1)}z^{-2(m-1)}, \tag{22}$$

where ℓ is in the range $0 \le \ell \le \lfloor\frac{M+1}{2}\rfloor$. The solution of the above equation is $G_\ell(z) = \tilde{G}_{2M-1-\ell}(z), \quad 0 \le \ell \le M-1$, which in turns implies that $H(z)$ and $F(z)$ are linear-phase polynomials.

Solution for $A(z)B(z) + C(z)D(z) = z^{-\alpha}$

Let $A(z) = \sum_{\ell=0}^{m-1} a_\ell z^{-\ell}$, $B(z) = \sum_{\ell=0}^{m-1} b_\ell z^{-\ell}$, $C(z) = \sum_{\ell=0}^{m-1} c_\ell z^{-\ell}$ and $D(z) = \sum_{\ell=0}^{m-1} d_\ell z^{-\ell}$. Suppose that α, $A(z)$ and $C(z)$ are given; this section will describe a method to solve for $B(z)$ and $D(z)$. The number of conditions and unknowns are $(2m-1)$ and $2m$ respectively. Therefore, in order for the solution to be unique, let b_0 be a fixed nonzero number. Consequently, from the first condition (corresponds to z^0), $d_0 = (a_0 b_0)/c_0$ (assuming that c_0 is nonzero). Thus, the number of conditions and unknowns, now, are both $(2m-2)$. In terms of a_ℓ, b_ℓ, c_ℓ and d_ℓ, one has

$$\begin{pmatrix} a_0 & 0 & \cdots & 0 & c_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & c_1 & c_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-2} & a_{m-3} & \cdots & a_0 & c_{m-2} & c_{m-3} & \cdots & c_0 \\ a_{m-1} & a_{m-2} & \cdots & a_1 & c_{m-1} & c_{m-2} & \cdots & c_1 \\ 0 & a_{m-1} & \cdots & a_2 & 0 & c_{m-1} & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1} & 0 & 0 & \cdots & c_{m-1} \end{pmatrix} \bullet$$

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \\ d_1 \\ d_2 \\ \vdots \\ d_{m-1} \end{pmatrix} = \mathbf{1} - \begin{pmatrix} a_1 b_0 + c_1 d_0 \\ a_2 b_0 + c_2 d_0 \\ \vdots \\ a_{m-1} b_0 + c_{m-1} d_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (23)$$

which can be solved if a_0, a_{m-1}, c_0 and c_{m-1} are all nonzero. Here $\mathbf{1}$ is the vector of zero elements except at the α position. Note that α only appears in the vector $\mathbf{1}$.

Design Procedure

1. Given m, M, α, s and ω_S, the stopband cutoff frequency of $H(z)$, design a lowpass filter.
2. Initialize $G_\ell(z)$ and $G_{M-1-\ell}(z)$, $0 \le \ell \le \lfloor \frac{M+1}{2} \rfloor$ from the coefficients of $H(z)$.
3. For each ℓ, find the corresponding polyphase filters $G_{2M-1-\ell}(z)$ and $G_{M+\ell}(z)$ using the technique in the above subsection. Form the new $H(z)$ which is the prototype of a PR variable-overall-delay cosine-modulated filter bank.
4. Minimize $\Phi = \int_{\omega_S}^{\pi} |H(e^{j\omega})|^2 d\omega$ using a minimization algorithm. The parameters are the coefficients of $G_\ell(z)$ and $G_{M-1-\ell}(z), 0 \le \ell \le \lfloor \frac{M+1}{2} \rfloor$. In other words, if Φ is not small enough, perturb $G_\ell(z)$ and $G_{M-1-\ell}(z)$ and repeat step 3 and 4. The total number of parameters is mM.
5. Once the optimized prototype filter $H(z)$ is found, the corresponding $F(z)$ is obtained from (19) and (6). Moreover, the analysis and synthesis filters can also be found using (1).

Example Fig. 2 plots the impulse response coefficients, $h(n)$, for the cases where $M = 10, m = 3$ and $\alpha = 1, 2$ and 4. When $\alpha = 1 (\alpha < m-1)$, $h(n)$ behaves as a typical minimum-phase impulse response [5], whereas it behaves as a typical maximum-phase impulse response [5] when $\alpha = 4 (\alpha > m-1)$. Thus, depending on whether α is smaller or larger than $m-1$, the corresponding $h(n)$ behaves as minimum or maximum-phase response. Of course, $h(n)$ is linear-phase if $\alpha = m-1$, as expected.

Fig. 3 shows the magnitude response plots of the prototype filter $H(z)$ for the above example, i.e., $M = 10, m = 3$ and $\alpha = 1, 2, 4$. The corresponding magnitude responses of the analysis filters $H_k(z)$ are plotted in Fig. 4. To demonstrate the variable overall-delay capability of the proposed filter banks, they are used to reconstruct the following signal

$$x(n) = n + 1, \quad 0 \le n \le 9.$$

The original input and the reconstructed output sequences for the above variable overall delay cosine-modulated filter banks are plotted in Fig. 5. Note the overall delay of $\hat{x}(n)$ for different values of α.

Topics for future work:

- In the paper, we assume that the matrix $\hat{\mathbf{C}}_F$ ($\hat{\mathbf{C}}$ of the synthesis filter bank) is the transpose of $\hat{\mathbf{C}}_H$ ($\hat{\mathbf{C}}$ of the analysis filter bank) because θ_k and γ_k are as in (2). Suppose that θ_k and γ_k in (1) are chosen differently, then $\hat{\mathbf{C}}_F \ne \hat{\mathbf{C}}_H^T$. What can we conclude from this choice and can the result here be extended to that case?
- Instead of $\alpha_\ell = \alpha$ and $\beta_\ell = 1$ as in $D_\ell(z)$, (see (20) and the text following it), let them be different for all ℓ. Could this choice yield a more general variable-overall-delay filter bank?

III. Conclusion

We present the variable-overall-delay PR cosine modulated filter bank. The overall delay of this system can take several values. Moreover, the conventional cosine-modulated filter bank is shown to be a special case of the proposed filter bank. An example is given to demonstate the theory.

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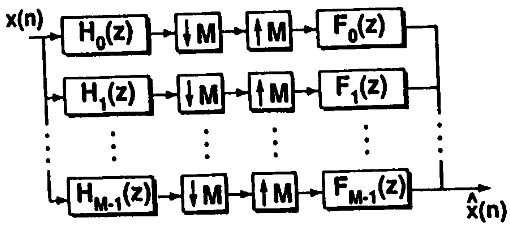

Fig.1. The M-channel maximally decimated filter bank.

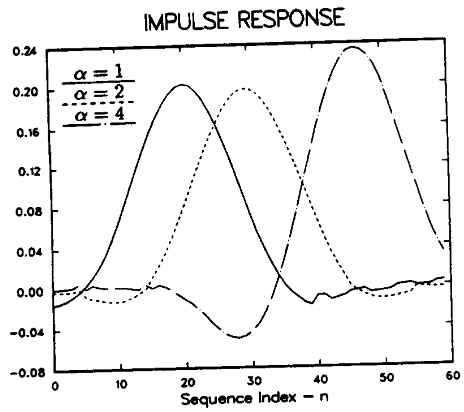

Fig.2. Plot of the impulse response $h(n)$ for the cases where $\alpha = 1, 2$ and 4.

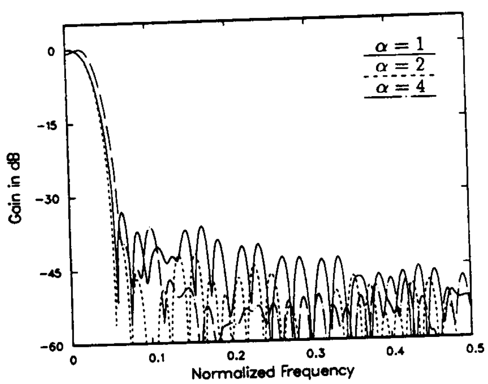

Fig. 3. Magnitude response plots for the prototype filter $H(z)$.

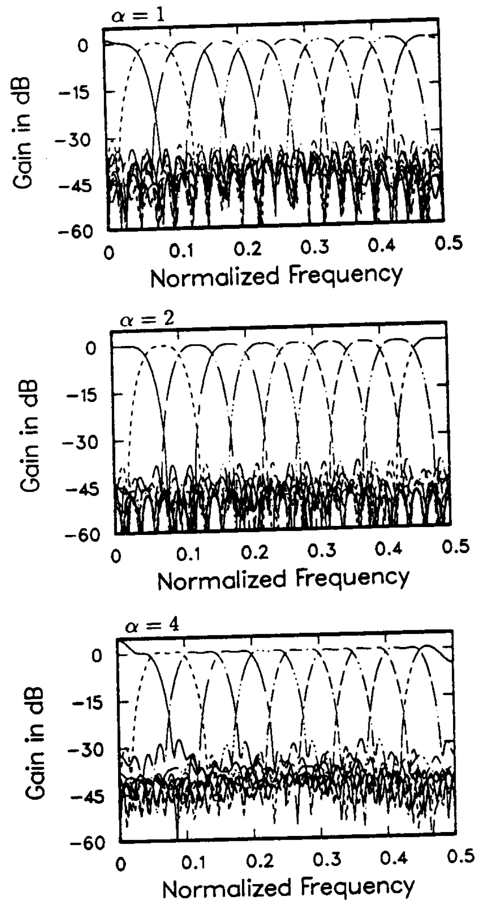

Fig. 4. Magnitude response plots for the analysis filters $H_k(z)$.

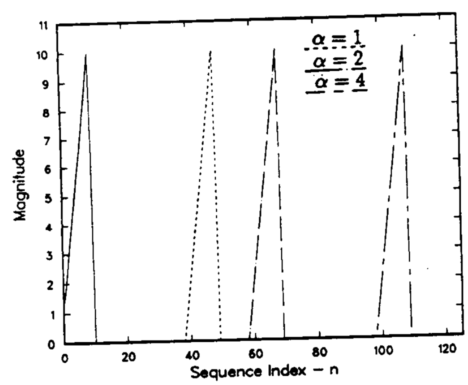

Fig. 5. Plot of the input signal x(n) and the reconstructed signal $\hat{x}(n)$. Note the corresponding delays for different values of α.